§7. The Variational Principles of Mechanics

Isaac Newton introduced

"vectorial mechanics"

$$\vec{F} = m\vec{a}$$

In a many-particle system (solid body, fluid) one has to determine the force on a particle exerted on it by <u>all</u> other particles

→ Newton introduces "action = reaction"

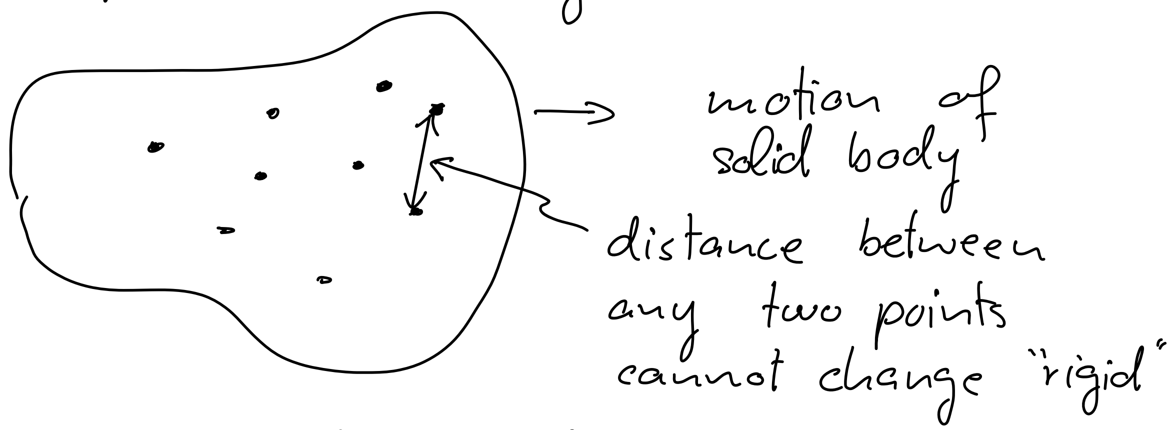
<u>But</u>: further assumptions on nature of forces have to be made

→ solution is not unique

Euler and Lagrange (~ 1750) introduced "analytical mechanics":

- particle is no longer an isolated unit but part of a "system"

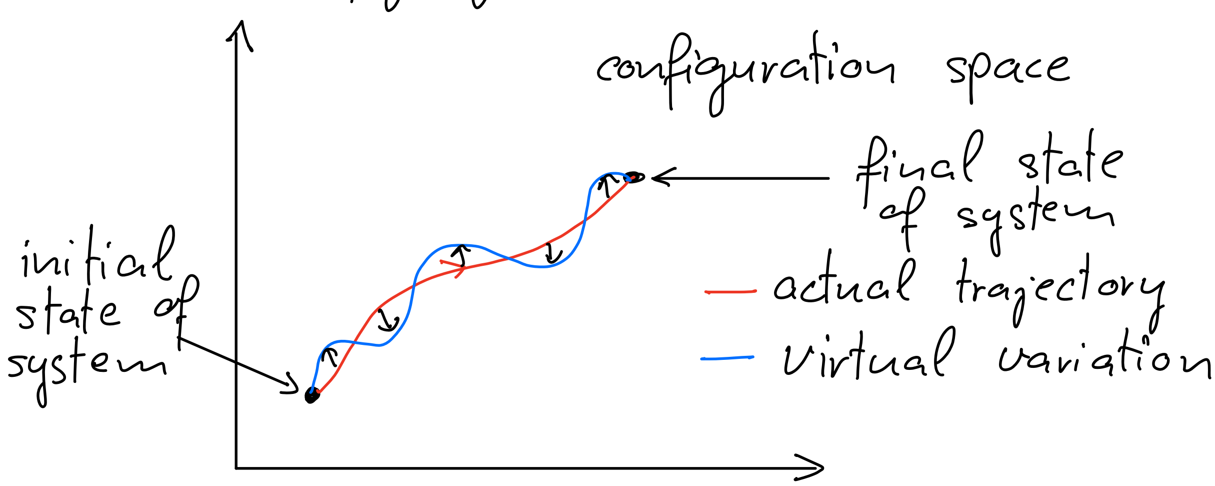
- incorporates "auxiliary conditions":



→ Knowledge of forces maintaining Kinematical conditions is not necessary

Similarly, the Kinematical condition during motion of fluid is that the volume of any portion must be preserved

- "Unifying principle":



Demand that a fundamental <u>scalar</u> quantity, known as the "action", remains stationary under displacements of trajectories

→ equations of motion of many-particle system follow!

Generalized coordinates:

Consider a system of N free particles with coordinates: x_i, y_i, z_i $(i=1,2,\dots,N)$

→ express in terms of new coordinates:

$$q_1, q_2, \dots, q_{3N}$$

→ determine these quantities as functions of time t

For example, we can make a coordinate transformation to polar coordinates:

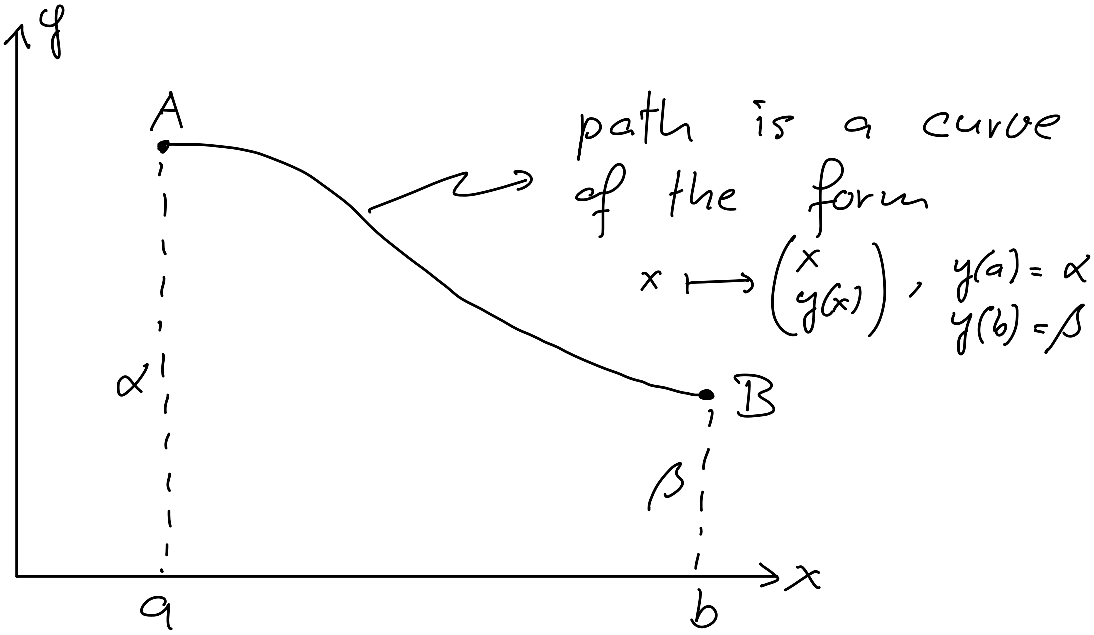
$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta \end{aligned} \xrightarrow{\text{in general}} \begin{aligned} x_1 &= f_1(q_1,\dots,q_{3N}) \\ &\vdots \\ z_N &= f_{3N}(q_1,\dots,q_{3N}) \end{aligned}$$

§7.1 Calculus of variations:

In order to formulate the equations of analytical mechanics, we first need to introduce variational problems

Example 1:

We wish to find a suitable plane curve along which a particle descends in the shortest possible time



We can determine the velocity from conservation of energy:

$$\frac{1}{2} m v(x)^2 = m g(\alpha - y(x)) \quad \text{or} \quad v(x) = \sqrt{2g(\alpha - y)}$$

The length of a small segment of the path is given by

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$
$$= dx \sqrt{1 + y'(x)^2}$$

Together with

$$v(t) = \frac{ds}{dt} \iff \frac{dt}{ds} = \frac{1}{v(s)}$$

we obtain for the total travel time T:

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)} ds$$
$$= \frac{1}{\sqrt{2g}} \int_a^b \frac{\sqrt{1+y'(x)^2}}{\sqrt{\alpha - y}} dx, \quad \text{set } F(y,y',x) := \frac{\sqrt{1+y'^2}}{\sqrt{2g(\alpha-y)}}$$

Among all possible functions $y(x)$ we want to find the particular one which yields the smallest possible value of T.

$\longrightarrow$ minimize "functional" $I := \int_a^b F(y,y',x)\, dx$

subject to the constraints $y(a) = \alpha$, $y(b) = \beta$.

General solution (for arbitrary $F(y, y', x)$):

Consider the function $y = f(x)$ which by hypothesis gives a stationary value for F

→ consider modification

$$\bar{f}(x) = f(x) + \varepsilon \phi(x) \quad , \quad \varepsilon \ll 1$$

arbitrary differentiable function

Then, at point x, we have:

$$\delta y = \bar{f}(x) - f(x) = \varepsilon \phi(x) \quad , \quad \delta x = 0$$

with constraints

$$[\delta f(x)]_{x=a} = 0 \, , \quad [\delta f(x)]_{x=b} = 0$$

Note that

- $\frac{d}{dx} \delta y = \frac{d}{dx}\left[\overline{f(x)} - f(x)\right] = \frac{d}{dx} \varepsilon \phi(x) = \varepsilon \phi'(x)$
- $\delta \frac{d}{dx} f(x) = \overline{f'(x)} - f'(x) = (y' + \varepsilon \phi') - y' = \varepsilon \phi'(x)$

$\longrightarrow \quad \frac{d}{dx} \delta y = \delta \frac{d}{dx} y$

Similarly, one can show

$$\delta \int_a^b F(x)\, dx = \int_a^b \delta F(x)\, dx$$

Now we compute

$$\delta F(y, y', x) = F(y + \varepsilon\phi, y' + \varepsilon\phi', x) - F(y, y', x)$$
$$= \varepsilon \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right)$$

$$\longrightarrow \delta \int_a^b F\,dx = \int_a^b \delta F\,dx = \varepsilon \int_a^b \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right) dx$$

A stationary $y = f(x)$ satisfies

$$\frac{\delta I}{\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right) dx = 0$$

Now perform integration by parts on second term:

$$\int_a^b \frac{\partial F}{\partial y'} \phi'\,dx = \underbrace{\left[\frac{\partial F}{\partial y'} \phi \right]_a^b}_{=0} - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \phi\,dx$$

$$\longrightarrow \frac{\delta I}{\varepsilon} = \int_a^b \underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right)}_{=: E(x)} \phi\,dx$$

$$\Longleftrightarrow \int_a^b E(x)\,\phi(x)\,dx = 0 \qquad (*)$$

for <u>arbitrary</u> $\phi(x)$!

Lemma 1:

Equation (*) can only be satisfied iff $E(x) = 0$ everywhere on $[a, b]$.

Proof:

Choose $\phi(x)$ s.th. it vanishes everywhere except for an arbitrarily small interval around $x = \xi$

$$\longrightarrow \quad 0 = E(\xi) \underbrace{\int_{\xi-\rho}^{\xi+\rho} \phi(x)\, dx}_{\neq 0} \quad \Rightarrow \quad E(\xi) = 0$$

$\square$

(*) + Lemma $\longrightarrow$ obtain differential eq.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

on entire interval $[a, b]$!

general form: $I = \int_{t_1}^{t_2} L(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)\, dt$

Demanding

$$\delta I = 0, \quad [\delta q_k(t)]_{t=t_1} = 0, \quad [\delta q_k(t)]_{t=t_2} = 0$$

gives

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \quad (t_1 \le t \le t_2), \; k = 1, \dots, n$$

"Euler – Lagrange equations"

§7.2 The principal of virtual work

- assume that there are forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ acting at points $P_1, P_2, \dots, P_n$ of a many-particle system
- assume that we perturb the system by "virtual" displacements of $P_1, P_2, \dots, P_n$ denoted by

$$\delta\vec{R}_1, \delta\vec{R}_2, \dots, \delta\vec{R}_n$$

(obeying kinematical constraints and being "reversible", i.e. $\delta\vec{R}_i \leftrightarrow -\delta\vec{R}_i$)

<u>Principal of virtual work</u>:

"The given mechanical system will be in equilibrium if, and only if, the total virtual work of all impressed forces vanishes:

$$\delta \overline{W} = \vec{F}_1 \cdot \delta\vec{R}_1 + \vec{F}_2 \cdot \delta\vec{R}_2 + \dots + \vec{F}_n \cdot \delta\vec{R}_n = 0$$

In cases where the forces F_i are derivable from a potential $V(q_1, \dots, q_n)$, that is

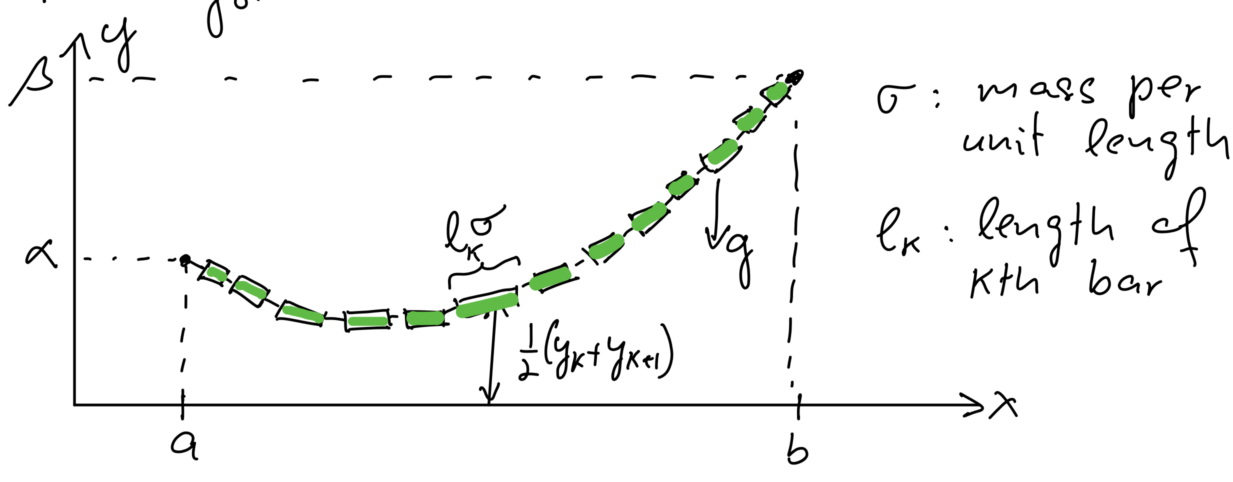
$\vec{F}_i = -\frac{\partial V}{\partial q_i}$, the principal of virtual work is equivalent to stating:

$$\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots + \frac{\partial V}{\partial q_n}\delta q_n = 0$$

If the equilibrium is "stable", the pot. energy must assume its minimum value, while in general it only is stationary

<u>Example 2:</u>

Consider a system of uniform rigid bars of constant cross-section, freely jointed at their end-points. The two free ends of the chain are suspended. Find the position of equilibrium of the system.



Potential energy:

$$V = g \sum_{k=0}^{n-1} m_k \frac{1}{2}(y_k + y_{k+1})$$

$$= \frac{\sigma g}{2} \sum_{k=0}^{n-1} (y_k + y_{k+1})\, \ell_k \qquad (1)$$

Contraints:

$$\ell_k^2 = (x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2 \qquad (2)$$

$$= (\Delta x_k)^2 + (\Delta y_k)^2, \quad k = 0, 1, \dots, n-1$$

$$= \text{fixed} \; !$$

<u>Task</u>: Find stationary value of potential in (1) for displacements for the y_k subject to the contraints (2)

<u>Question</u>: How do we incorporate contraints (2) into our variational problem ?

First, let us find the continuum form of the equations (1) and (2):

$$V = \int_{\tau_1}^{\tau_2} y \sqrt{x'^2 + y'^2}\, d\tau \quad (3) \qquad x'^2 + y'^2 = 1 \quad (4)$$

where we have assumed a parametric form for the resulting curve:

$$x = x(\tau), \qquad y = y(\tau)$$

Next, we will need

Lemma 2:

The stationary points of a functional

$$I = \int_{\tau_1}^{\tau_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)\, d\tau$$

subject to constraints

(*) $f_1(q_1, \dots, q_n, t) = 0, \dots, f_m(q_1, \dots, q_n, t) = 0$

can be obtained by variation of

$$I' = \int_{\tau_1}^{\tau_2} L' d\tau, \quad L' := L + \lambda_1 f_1 + \dots + \lambda_m f_m$$

with respect to $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$

Proof:

Variation with respect to λ_i:

$$\delta_{\lambda_i} I' = \int_{\tau_1}^{\tau_2} \Big(\frac{\partial L'}{\partial \lambda_i} - \frac{d}{d\tau} \underbrace{\frac{\partial L'}{\partial \dot{\lambda}_i}}_{=0} \Big) \delta\lambda_i = 0$$

$\rightarrow \dfrac{\partial L'}{\partial \lambda_i} = f_i = 0 \rightarrow$ automatically reproduces constraints (*) $\square$

Coming back to our problem, we see that equations (3) and (4) together with Lemma 2 result in finding the equilibrium with respect to the modified potential

$$\bar{V} = \int_{\tau_1}^{\tau_2} (y + \lambda)\sqrt{x'^2 + y'^2}\, d\tau$$

(Homework)

§7.3 D'Alembert's Principle

Definition 1 (The force of inertia):

Consider the fundamental law of motion of Newton:

$$m\vec{a} = \vec{F} \iff \vec{F} - m\vec{a} = 0 \quad (*)$$

We now define a vector $\vec{I}$ by

$$\vec{I} = -m\vec{a} \quad \text{"force of inertia"}$$

→ equation (*) becomes $F + I = 0$

→ have reduced dynamics to statics

That is, by adding the force of inertia to a system, we can treat it as a static system and find its equilibrium by applying the principl of virtual work!

Adding the force of inertia, we define the effective force $\vec{F}_k^e$: $\vec{F}_k^e = \vec{F}_k + \vec{I}_k$

D'Alembert's principle:

The total virtual work of the effective forces is zero for all reversible variations which satisfy the given kinematical conditions:

$$\sum_{k=1}^{N} \vec{F}_k^e \cdot \delta\vec{R}_k = \sum_{k=1}^{N} (\vec{F}_k - m_k \vec{a}_k) \cdot \delta\vec{R}_k = 0$$

$$\Leftrightarrow \quad \delta V + \underbrace{\sum m_k \vec{a}_k \cdot \delta\vec{R}_k}_{= -\delta \overline{W^i}} = 0 \qquad (**)$$

$= -\delta\overline{W^i}$ cannot be rewritten as variation of scalar function

Using $\sum m_k \ddot{\vec{R}}_k \cdot d\vec{R}_k = \sum m_k \ddot{\vec{R}}_k \cdot \dot{\vec{R}}_k \, dt$

$$= \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\sum m_k \dot{\vec{R}}_k^2}_{=:T}\Big)dt = dT$$

we see that $(**)$ is equivalent to

$$dV + dT = d(V+T) = 0 \Rightarrow T+V = \text{const.} = E$$

"conservation of energy"